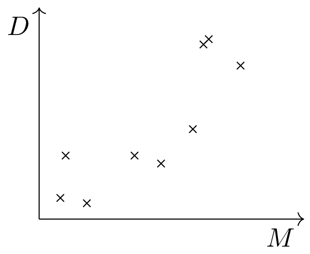
2801. Prove that, if two monic quadratic graphs $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ do not intersect, then they must be translations of one another parallel to the y axis.

2802. A seismologist is studying the effect of earthquakes on leakage in pipes. She compares the magnitude M of earthquakes to the leakage L in the 24 hours following a quake. The variable M is measured on a logarithmic scale, which means that $M = \ln R$, where R is the raw magnitude.



The seismologist finds that the curve of best fit is $D = 1.523e^{2.049M}$. Determine whether this is consistent with a model in which the relationship between the raw magnitude R and water leakage D is linear.

2803. Variables x and y satisfy

$$x^2\sqrt{y} + x = \frac{20}{\sqrt{y}}.$$

Find y in terms of x. Your answer should be given in two cases, one for $x < 0$ and one for $x > 0$.

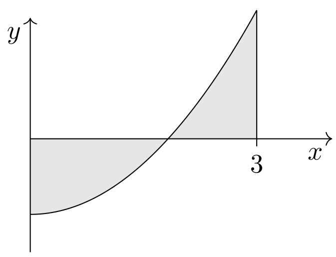
2804. A hand of four cards is dealt from a standard deck. Find the probability that

(a) all four are the same suit,

(b) two are black and two are red.

2805. Disprove the following claim: "For a function to be invertible over the domain $\mathbb{R}$, it must either be increasing everywhere or decreasing everywhere."

2806. The graph below shows a part of the parabola $y = ax^2 - 1$, with two regions of equal area shaded.



(a) Determine a.

(b) Show that the area shaded is $\frac{4\sqrt{3}}{3}$.

(c) Write down the value of $\displaystyle\int_0^3 ax^2 - 1\,dx$.

2807. Find and classify all stationary points of

$$y = \frac{4}{\sqrt{x}} + \sqrt{x}.$$

2808. Find simplified expressions for the following sets, without using the set-minus notation $* \setminus *$:

(a) $A \setminus (B \setminus A)$,

(b) $A \setminus (B \setminus A')$,

(c) $A \setminus (B' \setminus A')$.

2809. Four integers $a < b < c < d$ are such that the mean of a and d is the same as that of b and c. Prove that this information is necessary, but not sufficient, for a, b, c, d to be an arithmetic sequence.

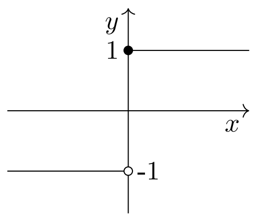
2810. Show that there are (x, y) points which satisfy

$$\begin{aligned} &x^2 > y^3, \\ &x^2 + (y-2)^2 = 2. \end{aligned}$$

2811. State, with a reason, whether the following holds: "In a binomial hypothesis test with $H_1 : p \neq p_0$, if $\mathbb{P}(X \leq k) < 0.01$, then, at the 1% significance level, k lies in the critical region for the test."

2812. Prove that $a^{\ln b} \equiv b^{\ln a}$.

2813. A function f, defined on $\mathbb{R}$, has the graph depicted.



Sketch the following graphs:

(a) $y = |\mathrm{f}(x)|$,

(b) $|y| = \mathrm{f}(x)$,

(c) $|y| = |\mathrm{f}(x)|$.

2814. Two vectors are defined, for $x \in \mathbb{R}$, as

$$\begin{aligned} \mathbf{p} &= x\mathbf{i} + (x+2)\mathbf{j}, \\ \mathbf{q} &= (x-1)\mathbf{i} + (x+3)\mathbf{j}. \end{aligned}$$

Show that $\mathbf{p}$ and $\mathbf{q}$ cannot be perpendicular.

2815. From a small population of a hundred, a sample of five is taken. Assuming that exactly 75% of the population is below the upper quartile, find the probability that at least one of the sample of five is at or above the upper quartile.

2816. A particle moves on an elliptical trajectory, with coordinates (x, y) given as

$$x = 3\cos t,$$
$$y = 2\sin t.$$

(a) Find the Cartesian equation of the trajectory.
(b) Show that the particle is moving fastest when it is closest to the origin.

2817. Prove that, for non-constant variables a, b, c, it is not possible to have $a \propto b^2$, $b \propto c^2$, and $c \propto a^2$.

2818. Two sets of data have means satisfying $\bar{x}_1 < \bar{x}_2$ and standard deviations satisfying $s_1 < s_2$. State, with a reason, whether each of the following statements, concerning the mean $\bar{x}_c$ and standard deviation s_c of the combination of the two sets, is necessarily true:

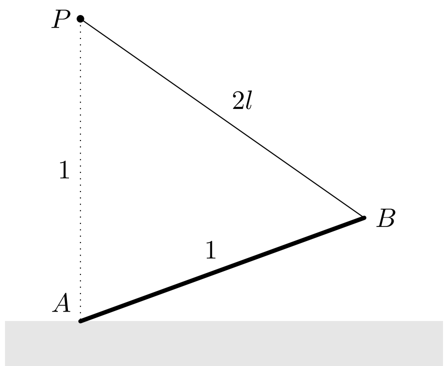
(a) $\bar{x}_1 < \bar{x}_c < \bar{x}_2$,
(b) $s_1 < s_c < s_2$.

2819. Describe the single transformation which takes the graph $y = x^2 - 2x$ onto the graph $y = x^2 - 1$.

2820. Find the probability that two numbers chosen at random on the interval $[0, 1]$ add to more than 1.

2821. Show that $\mathrm{f}(x) = (6\cos x + 7\sin x + 9)^{\frac{3}{4}}$ is not well defined on $\mathbb{R}$.

2822. A uniform rod AB of mass m and length 1 m is attached to a point P, 1 m above rough horizontal ground, by a light string of length $2l$ metres. The string is attached at B under tension T, and point A rests on the ground vertically below P.



(a) Draw a force diagram for the rod.
(b) Show that $T = mgl$.

2823. A family of circles is given by

$$(x - r_n)^2 + y^2 = r_n^2.$$

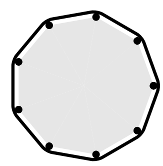
The radii $r_n > 0$ form an arithmetic progression whose common difference is equal to its first term. Sketch the family of circles.

2824. On a circle of (fixed) radius r, and with θ defined in radians, describe the quantity $r\frac{d\theta}{dt}$, where t is time.

2825. A graph, whose equation is $\mathrm{f}(x)+\mathrm{g}(y) = 1$ for some functions f and g, is reflected in the x axis. Write down the equation of the transformed graph.

2826. Eliminate t from $x\sin t = 1$, $y\tan t = 1$. Give your answer in simplified Cartesian form.

2827. A loop of light string is passed, under tension T, around n smooth pegs, which lie at the vertices of a regular n-gon of side length 1.



Show that the magnitude of the force exerted on each peg by the string is $2T\sin\frac{\pi}{n}$.

2828. An integral equation is given as

$$\int y\,dx = 3y + c.$$

By first differentiating both sides, solve to find a general formula for y in terms of x.

2829. Either prove or disprove the following statement: "If $\mathrm{f}(k) = \mathrm{f}^{-1}(k)$, then k is a fixed point of the function f."

2830. A positive polynomial curve of even degree has two single roots at $x = a$ and $x = b$ and a double root at $x = c$, where $a < b < c$.

(a) Show that there must be a stationary point in each of the intervals (a, b) and (b, c).
(b) Show that there must be a point of inflection in (b, c), but not necessarily in (a, b).

2831. Solve for x in $\displaystyle\sum_{i=1}^{\infty} x^i\left(1 + x + x^2\right) = \frac{13}{18}$.

2832. Three couples sit down at random around a round table. Find the probability that everyone ends up sitting opposite their partner.

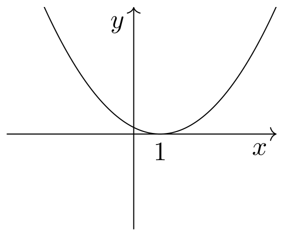
2833. Disprove the following statement:

$$\mathrm{f}(x) \not\equiv \mathrm{g}(x) \implies \int_0^x \mathrm{f}(t)\,dt \not\equiv \int_0^x \mathrm{g}(t)\,dt.$$

2834. Prove that, if tangent lines are drawn to the parabola $y = ax^2 + bx + c$ at $x = \pm q$, then they meet on the y axis.

2835. You are given that $\mathbf{p}$ and $\mathbf{q}$ are perpendicular, and also that $2\mathbf{p}+3\mathbf{q}$ and $-3\mathbf{p}+2\mathbf{q}$ are perpendicular. Prove that $|\mathbf{p}| = |\mathbf{q}|$.

2836. The graph is $y = \mathrm{h}'(x)$, where h is a cubic function.



You are given that $\mathrm{h}(0) = -1$ and $\mathrm{h}(2) = 7$. Find the stationary point of $y = \mathrm{h}(x)$.

2837. Prove that $\dfrac{1+\sin\theta}{1-\sin\theta} \equiv (\sec\theta + \tan\theta)^2$.

2838. Prove that the only function which is both even and odd is the zero function.

2839. Two particles are moving in one dimension. The particles, which are in the same place at time $t = 0$, have velocities given, for $t \geq 0$, by

$$v_a = \frac{1}{1+t}, \qquad v_b = \frac{1}{1+2t}.$$

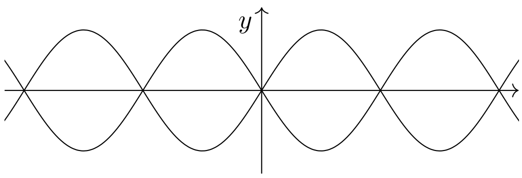
Show that the particles do not meet again.

2840. Simultaneous equations are as follows:

$$xy = a^2,$$
$$(\log_a x)^2 - (\log_a y)^2 = 1.$$

Solve these equations, giving x and y in terms of the positive constant a.

2841. The graph of a relation is



State whether any of the following could be the equation generating the graph:

(a) $y^2 = \cos^2 x$,

(b) $|y| = \cos^2 x$,

(c) $y^2 = \sin^2 x$,

(d) $|y| = \sin^2 x$.

2842. Show that the cubic equation $4x^3 + 7x - 8 = 0$ has exactly one real root.

2843. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$-1 \leq x + y \leq 1,$$
$$-1 \leq x - y \leq 1.$$

2844. You are given that the variance of a sample $\{x_i\}$ is $s^2 = 7.281$. State, with a reason, whether either of the following quantities can, with the information given, be guaranteed to be larger than the other:

(a) $\sum |x_i - \bar{x}|$

(b) $\sum (x_i - \bar{x})^2$.

2845. Show that the line $y = 2x - e$ is tangent to the curve $y = x \ln x$.

2846. Find the probability that, if the seven letters of the word ANAGRAM are arranged at random, the original word is spelled out.

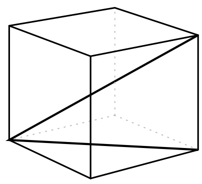
2847. Graph G is defined by $y = |x+2| + |x-2| - 3$.

(a) Find the coordinates of the two vertices of G.

(b) Determine the three integer values taken by the gradient of G.

(c) Hence, sketch the curve.

2848. In a cube, a face diagonal and a space diagonal are constructed, starting from the same vertex.



Find the angle between them, giving your answer in the exact form $\arccos k$.

2849. Verify that $y = \cot x$ satisfies the DE

$$y^2 + \frac{dy}{dx} = -1.$$

2850. A graph $y = \mathrm{g}(x)$, of even degree, has exactly two stationary points. Prove that exactly one of these stationary points must also be a point of inflection.

2851. Show that no (x, y) points simultaneously satisfy the inequalities $y > 2 + |x-4|$ and $x < 7 - y^2$.

2852. It can be shown that, for $\theta \in (0, \pi/2)$,

$$1 < \frac{\theta}{\sin\theta} < \sec\theta.$$

Explain how this can be used to prove that, for small angles, $\sin\theta \approx \theta$.

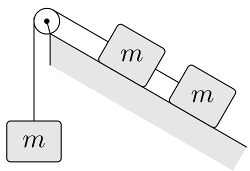
2853. State, with a reason, whether these hold:

(a) $a+b \in (0,1) \implies a^2+b^2 \in (0,1)$,

(b) $a^2+b^2 \in (0,1) \implies a+b \in (0,1)$.

2854. Show that the curves $x+y=(x-y)^2+2$ and $y=x^2+x$ are tangent.

2855. Three masses are connected by light, inextensible strings, one of which is passed over a smooth, light, fixed pulley as shown in the diagram. The slope is smooth, at inclination 30°.



With the system in equilibrium, the shorter string is cut. Find, to 3sf, the time taken for the two masses on the slope to separate by 50 cm.

2856. Sketch the graph $y=\log_2 x - \log_4 x$.

2857. In circle geometry, the *sagitta* is a line segment which joins the midpoint of a chord AB to the midpoint of the minor arc AB. It is also known as the *depth of an arc*.

(a) On a circle, sketch this information, labelling the length of the sagitta as x and the length of the chord as y.

(b) Prove that the radius of the circle is given by

$$r=\frac{y^2}{8x}+\frac{x}{2}.$$

2858. Solve the equation $\cos^2\theta=\cos 2\theta+1$, giving all values $\theta\in[-\pi,\pi]$.

2859. A polynomial $x\mapsto \mathrm{g}(x)$ is increasing for all $x\in\mathbb{R}$. Prove that g must be of odd degree.

You may assume that every polynomial equation of odd degree has at least one real root.

2860. An inequality is given, for constants $a<b$, as

$$\frac{x-a}{x-b}>1.$$

By considering the magnitudes of numerator and denominator, or otherwise, solve the inequality.

2861. The graph $\mathrm{f}(x)+\mathrm{f}(y)=k$, where f is a function, is translated by 4 units in the positive y direction. Write down the equation of the transformed graph.

2862. Two particles are moving in 1D, for $t\in[-10,10]$. Their position vectors are given by

$$\begin{aligned}\mathbf{r}_1&=(2t+1)\mathbf{i}+\mathbf{j},\\ \mathbf{r}_2&=(1-t)\mathbf{i}+4t\mathbf{j}.\end{aligned}$$

(a) Formulate an expression for d^2, the squared distance between the particles at time t.

(b) Hence, find the period of time for which the particles are closer to each other than $\sqrt{18}$, giving your answer in interval set notation.

2863. Show that the best linear approximation g to the function $\mathrm{f}: x\mapsto \sin(x^2)$ at $x=\sqrt{\pi}$ is given by $\mathrm{g}(x)=2\pi-2\sqrt{\pi}x$.

2864. Consider the expression $4^x+15^x-3^x-20^x$.

(a) Factorise this expression.

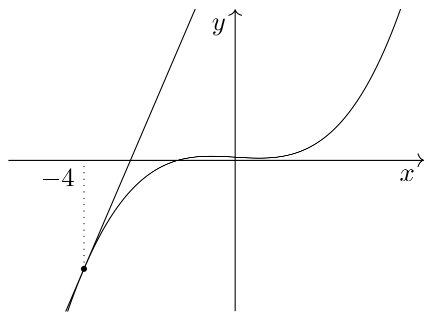
(b) Hence, show that the expression is only zero when $x=0$.

2865. Show that $\displaystyle\int_a^{2a}\frac{x^3-1}{x^2}\,dx \equiv \frac{3a^3-1}{2a}$.

2866. Prove that, if exactly three forces are acting on an object which is in equilibrium, then the lines of action of those three forces must be concurrent.

2867. In this question, do not use a polynomial solver.

The diagram shows the cubic $y=5x^3-6x+8$ and the tangent T at $x=-4$. The tangent crosses the cubic again for some large positive x.



(a) Show that T has equation $y=234x+648$.

(b) Determine the x coordinate of the point at which T intersects the curve again.

2868. Solve the equation $5\ln x+4=\dfrac{1}{\ln x}$.

2869. A curve is defined by the following parametric equations, in which t takes all real values:

$$\begin{aligned}x&=t^3,\\ y&=1-t^2.\end{aligned}$$

Determine the area of the region enclosed by this curve and the x axis.

2870. Prove that no polynomial function is periodic.

2871. True or false?

(a) $x, y \in (0, 1) \implies \dfrac{x}{y} \in (0, \infty)$,

(b) $x, y \in (0, \infty) \implies \dfrac{x}{y} \in (0, \infty)$,

(c) $x, y \in [0, 1] \implies \dfrac{x}{y} \in [0, \infty)$.

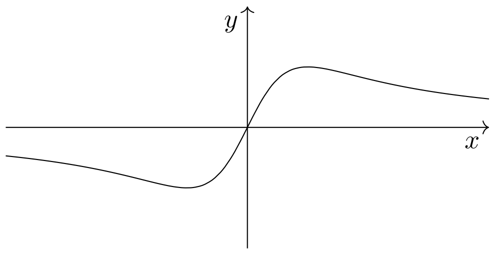
2872. Cubics $y = \mathrm{f}(x)$ and $x = \mathrm{g}(y)$ are being drawn. Let n be the number of distinct points of intersection. Sketch examples with

(a) the greatest n,

(b) the least n.

2873. Solve $\cos^5\theta = -\cos^3\theta$ for $\theta \in [-180, 180°]$.

2874. Show that $y = \dfrac{x}{x^2+4}$ has the behaviour depicted:

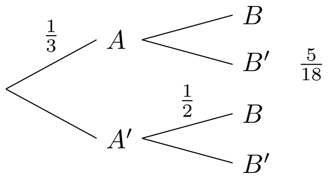


2875. Determine the set of values of m for which the graph $y = m^2x^2 + (m-1)\,x + 4$ has exactly two intersections with the x axis. Give your answer in set notation.

2876. It is given that x^2y^3 is constant. Find the ratio

$$\left.\frac{dy}{dx}\right|_{x=8} : \left.\frac{dy}{dx}\right|_{x=1}$$

2877. Events A and B have probabilities as represented on the following tree diagram:



Represent this information on a Venn diagram, giving the probabilities of all four regions.

2878. Show that, if $y = \ln\left(x + \sqrt{1+x^2}\right)$, then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}.$$

2879. Exactly four non-zero forces act on an object. One of these forces is perpendicular to each of the other three. Explain whether it is possible for the object to remain in equilibrium.

2880. A curve is defined by $y = x^3 - x^2 + 4$. At the point with y coordinate 8, a tangent is drawn. Show that this tangent crosses the x axis at $x = 1$.

2881. Find the coefficient of x^7 in $\left(1 + x + x^2\right)^5$.

2882. The inequalities $\mathrm{f}(x) \geq 0$ and $\mathrm{g}(x) \geq 0$ are solved simultaneously, giving solution set $S = [0, \infty)$. State, with a reason, whether the following facts are true, concerning the simultaneous solution set T of the two inequalities $\mathrm{f}(x) < 0$ and $\mathrm{g}(x) < 0$:

(a) T is $(-\infty, 0)$,

(b) T is a subset of $(-\infty, 0)$,

(c) T contains $(-\infty, 0)$.

2883. If $\dfrac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) = 0$, show that $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{y}{x}$.

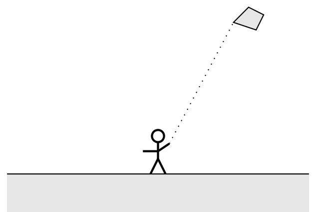
2884. State, giving your reasoning, which if any of the implications $\implies$, $\impliedby$, $\iff$ links the following statements concerning a polynomial function f:

① $\mathrm{f}(x)$ has a factor of $(x - a)$,

② $\mathrm{f}(x)$ has a double root at $x = a$.

2885. Show that $y = \sin^2(\arccos x)$ is part of a parabola.

2886. A boy, mass 40 kg, is flying a kite, mass 0.1 kg. The string of the kite is light and inextensible. The wind is exerting a sideways force of 30 N on the kite, whose string is angled at 20° to the vertical. The boy is at rest.



(a) Find the tension in the string, and also the lift exerted on the kite by the air.

(b) Explain how you know that the ground must be rough.

(c) The coefficient of friction between the boy's feet and the ground is μ. Determine the least possible value of μ.

2887. From a committee of thirty, a chairperson, a pair of secretaries and three adjutants are to be chosen. Find the number of different ways of doing this.

2888. True or false?

(a) $y = x^6 + x^3$ has a line of symmetry,

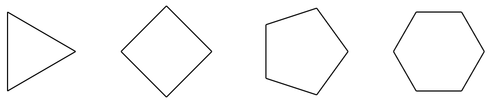
(b) $y = x^6 + x^4$ has a line of symmetry,

(c) $y = x^6 + x^5$ has a line of symmetry.

2889. A quartic in x is given as $x^4 - 10k^2x^2 + 9k^4 = 0$, for some constant $k \neq 0$. Show that this equation has four roots in arithmetic progression.

2890. Find the probability that five consecutive rolls of a die yield a strictly increasing sequence.

2891. The regular n-gons, for $n = 3, 4, 5, 6$ are



This set of n-gons is to be arranged in the (x, y) plane such that

① each vertex has non-negative y coordinate,
② each n-gon has a vertex at the origin,
③ no point (x, y) lies in the interior of more than two n-gons.

Determine whether or not this is possible.

2892. Show that no integers n satisfy

$$^n\mathrm{C}_3 + {}^n\mathrm{C}_5 = 2 \times {}^n\mathrm{C}_4.$$

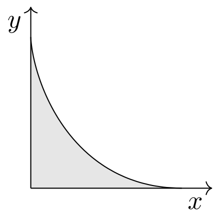
2893. Write the following in simplified interval notation:

$$\{x \in \mathbb{R} : x^3 - 4x > 0\} \cup \{x \in \mathbb{R} : x^2 - 1 \leq 0\}.$$

2894. A parametric graph is defined by the equations

$$x = t^3, \qquad y = (t^2 - 1)^2.$$

This graph is shown below, with a shaded region enclosed by the curve and the coordinate axes:



Find the area of the shaded region, using

$$A = \int_{t_1}^{t_2} y \frac{dx}{dt}\, dt.$$

2895. Show that the area of the region of the (x, y) plane defined by the following simultaneous inequalities is very close to $\sqrt{2}/2$.

$$x^2 + y^2 < 1,$$
$$2x + 1 < y.$$

2896. Prove that the exponential function $x \mapsto e^x$ is not equivalent to any polynomial function

$$x \longmapsto a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0.$$

2897. Show that $y = -x$ is tangent to $y = \dfrac{1}{1 + \frac{1}{x} + \frac{1}{x^2}}$.

2898. One of the following statements is true; the other is not. Identify and disprove the false statement.

- $\sec\theta = \sqrt{1+k} \implies \tan^2\theta = k$,
- $\sec\theta = \sqrt{1+k} \impliedby \tan^2\theta = k$.

2899. A *cycloid* is defined by the parametric equations $x = t - \sin t$, $y = 1 - \cos t$, for $t \in \mathbb{R}$. Verify that these equations satisfy the differential equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{2}{y} - 1.$$

2900. "The curves $y = x^2 + a$ and $x = y^2 + a$ are tangent for exactly one value $a \in \mathbb{R}$." True or false?

——— End of 29th Hundred ———